# Exercise 11

In Exercises 6 through 11, use the formal method, involving an infinite series of residues and illustrated in Examples 2 and 3 in Sec. 89, to find the function $f(t)$ that corresponds to the given function $F(s)$.

$$F(s) = \frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} \quad (0 < x < 1),$$

$$\text{where } \omega > 0 \text{ and } \omega \neq \omega_n = \frac{(2n-1)\pi}{2} \quad (n = 1, 2, \ldots).$$

$$\textit{Ans. } f(t) = \frac{\sin\omega x \sin\omega t}{\omega^2\cos\omega} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\omega_n}\cdot\frac{\sin\omega_n x\sin\omega_n t}{\omega^2-\omega_n^2}.$$

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## Solution

The inverse Laplace transform of the given function for $F(s)$ is defined by the Bromwich integral,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds,$$

where $\gamma$ is a real constant chosen such that all singularities of the integrand lie to the left of the infinite vertical line $(\gamma - i\infty, \gamma + i\infty)$ in the complex plane. They occur where the denominator is equal to zero.

$$s(s^2+\omega^2)\cosh s = 0$$

$$s = 0 \quad \text{or} \quad s^2+\omega^2 = 0 \quad \text{or} \quad \cosh s = 0$$

$$s = \pm i\omega \quad \text{or} \quad \cos is = 0$$

$$is = \frac{1}{2}(2k-1)\pi, \quad k = 0, \pm1, \pm2, \ldots \quad \rightarrow \quad \begin{cases} s_n = \dfrac{i\pi}{2}(2n-1), & n = 1, 2, \ldots \\ \overline{s_n} = -\dfrac{i\pi}{2}(2n-1), & n = 1, 2, \ldots \end{cases}$$

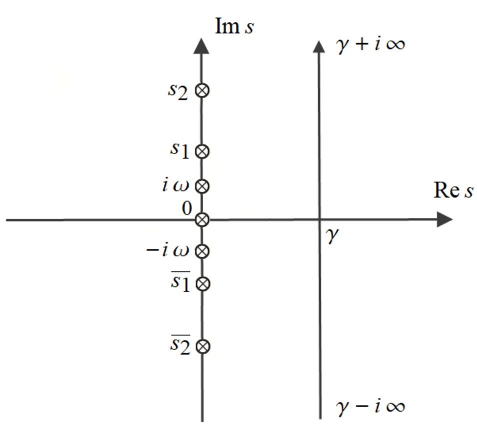


Figure 1: This is the complex plane with the singularities of the integrand marked as well as the vertical line $(\gamma - i\infty, \gamma + i\infty)$. $i\omega$ and $-i\omega$ are not necessarily where they appear in the figure.

The integral is evaluated by considering a closed loop integral in the complex plane containing this vertical line and then applying the Cauchy residue theorem to get an equation, allowing us to solve for it. Let the vertical line loop around to the bottom by a semicircular arc to the left as illustrated in Figure 2 so that the integral is positively oriented.

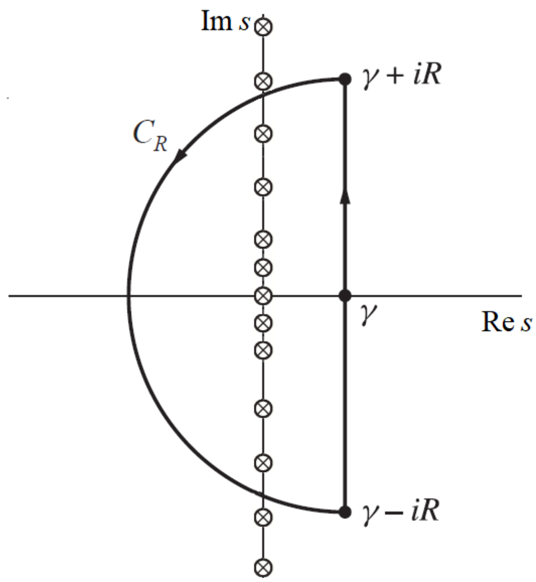


Figure 2: This is the closed loop that will be considered to calculate the inverse Laplace transform.

Now that the integration path is closed, the Cauchy residue theorem can be applied, which states that the integral over this path is equal to $2\pi i$ times the sum of the residues inside the loop.

$$\oint_C e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds = 2\pi i\left[\operatorname*{Res}_{s=0} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \operatorname*{Res}_{s=i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \sum_n \operatorname*{Res}_{s=s_n} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \sum_n \operatorname*{Res}_{s=\overline{s}_n} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\right]$$

Expand the left side.

$$\int_{\gamma-iR}^{\gamma+iR} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds + \int_{C_R} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds = 2\pi i\left[\operatorname*{Res}_{s=0} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \operatorname*{Res}_{s=i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \sum_n \operatorname*{Res}_{s=s_n} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \sum_n \operatorname*{Res}_{s=\overline{s}_n} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\right]$$

In the limit as $R \to \infty$ all the singularities become enclosed, making the series on the right side infinite. Also, the integral over $C_R$ tends to zero. Proof for this statement will be given at the end.

$$\begin{aligned}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds = 2\pi i\left[\operatorname*{Res}_{s=0} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \operatorname*{Res}_{s=i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\right. \\ + \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} \\ \left. + \sum_{n=1}^{\infty}\operatorname*{Res}_{s=s_n} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} + \sum_{n=1}^{\infty}\operatorname*{Res}_{s=\overline{s_n}} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\right]\end{aligned}$$

The residues at $s=i\omega$, $s=-i\omega$, $s=s_n$, and $s=\overline{s_n}$ are calculated by

$$\begin{aligned}\operatorname*{Res}_{s=i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= \frac{p(i\omega)}{q'(i\omega)} \\ \operatorname*{Res}_{s=-i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= \frac{p(-i\omega)}{q'(-i\omega)} \\ \operatorname*{Res}_{s=s_n} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= \frac{p(s_n)}{q'(s_n)} \\ \operatorname*{Res}_{s=\overline{s_n}} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= \frac{p(\overline{s_n})}{q'(\overline{s_n})},\end{aligned}$$

where

$$\begin{aligned}p(s) &= e^{st}\sinh(xs) \\ q(s) &= s(s^2+\omega^2)\cosh s \quad\rightarrow\quad q'(s) = (3s^2+\omega^2)\cosh s + s(s^2+\omega^2)\sinh s.\end{aligned}$$

We have

$$\begin{aligned}\frac{p(i\omega)}{q'(i\omega)} &= e^{i\omega t}\frac{\sinh i\omega x}{-2\omega^2\cosh i\omega} = -\frac{i}{2}e^{i\omega t}\frac{\sin\omega x}{\omega^2\cos\omega} \\ \frac{p(-i\omega)}{q'(-i\omega)} &= e^{-i\omega t}\frac{\sinh(-i\omega x)}{-2\omega^2\cosh(-i\omega)} = \frac{i}{2}e^{-i\omega t}\frac{\sin\omega x}{\omega^2\cos\omega} \\ \frac{p(s_n)}{q'(s_n)} &= \exp\left[\frac{i\pi}{2}(2n-1)t\right]\frac{\sinh\left[\frac{i\pi}{2}(2n-1)x\right]}{\left[\frac{i\pi}{2}(2n-1)\right]\left[-\frac{\pi^2}{4}(2n-1)^2+\omega^2\right]\sinh\left[\frac{i\pi}{2}(2n-1)\right]} \\ &= \exp(i\omega_n t)\frac{\sin\left[\frac{\pi}{2}(2n-1)x\right]}{i\omega_n(-\omega_n^2+\omega^2)\sin\left[\frac{\pi}{2}(2n-1)\right]} \\ &= \frac{(-1)^{n+1}}{i\omega_n(\omega^2-\omega_n^2)}e^{i\omega_n t}\sin\omega_n x \\ \frac{p(\overline{s_n})}{q'(\overline{s_n})} &= \exp\left[-\frac{i\pi}{2}(2n-1)t\right]\frac{\sinh\left[-\frac{i\pi}{2}(2n-1)x\right]}{\left[-\frac{i\pi}{2}(2n-1)\right]\left[-\frac{\pi^2}{4}(2n-1)^2+\omega^2\right]\sinh\left[-\frac{i\pi}{2}(2n-1)\right]} \\ &= \exp(-i\omega_n t)\frac{\sin\left[\frac{\pi}{2}(2n-1)x\right]}{-i\omega_n(-\omega_n^2+\omega^2)\sin\left[\frac{\pi}{2}(2n-1)\right]} \\ &= -\frac{(-1)^{n+1}}{i\omega_n(\omega^2-\omega_n^2)}e^{-i\omega_n t}\sin\omega_n x\end{aligned}$$

using the fact that $\sin\left[\frac{\pi}{2}(2n-1)\right] = (-1)^{n+1}$. Consequently,

$$\begin{aligned}
\operatorname*{Res}_{s=i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= -\frac{i}{2}e^{i\omega t}\frac{\sin\omega x}{\omega^2\cos\omega} \\
\operatorname*{Res}_{s=-i\omega} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= \frac{i}{2}e^{-i\omega t}\frac{\sin\omega x}{\omega^2\cos\omega} \\
\operatorname*{Res}_{s=s_n} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= \frac{(-1)^{n+1}}{i\omega_n(\omega^2-\omega_n^2)}e^{i\omega_n t}\sin\omega_n x \\
\operatorname*{Res}_{s=\overline{s}_n} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= -\frac{(-1)^{n+1}}{i\omega_n(\omega^2-\omega_n^2)}e^{-i\omega_n t}\sin\omega_n x.
\end{aligned}$$

The residue at $s=0$ cannot be calculated in the same way because $p(0)=0$. The series expansion of the integrand about $s=0$ has to be considered instead, and the residue will be the coefficient of $1/s$.

$$\begin{aligned}
e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} &= \left[1+(st)+\frac{1}{2}(st)^2+\cdots\right]\frac{1}{s(s^2+\omega^2)}\frac{(xs)+\frac{(xs)^3}{6}+\frac{(xs)^5}{120}+\cdots}{1+\frac{1}{2}s^2+\frac{1}{24}s^4+\cdots} \\
&= \left[1+(st)+\frac{1}{2}(st)^2+\cdots\right]\frac{1}{s^2+\omega^2}\frac{x+\frac{x^3}{6}s^2+\frac{x^5}{120}s^4+\cdots}{1+\frac{1}{2}s^2+\frac{1}{24}s^4+\cdots} \\
&= \left[1+(st)+\frac{1}{2}(st)^2+\cdots\right]\frac{1}{s^2+\omega^2}\left[x+\left(\frac{x^3}{6}-\frac{x}{2}\right)s^2+\left(\frac{x^5}{120}-\frac{x^3}{12}+\frac{5x}{24}\right)s^4+\cdots\right] \\
&= \frac{1}{s^2+\omega^2}\left[x+(xt)s+\left(\frac{1}{2}t^2+\frac{x^3}{6}-\frac{x}{2}\right)s^2+\cdots\right] \\
&= \frac{1}{\omega^2\left[1-\left(-\frac{s^2}{\omega^2}\right)\right]}\left[x+(xt)s+\left(\frac{1}{2}t^2+\frac{x^3}{6}-\frac{x}{2}\right)s^2+\cdots\right] \\
&= \frac{1}{\omega^2}\left(1-\frac{s^2}{\omega^2}+\frac{s^4}{\omega^4}-\cdots\right)\left[x+(xt)s+\left(\frac{1}{2}t^2+\frac{x^3}{6}-\frac{x}{2}\right)s^2+\cdots\right] \\
&= \frac{x}{\omega^2}+\frac{xt}{\omega^2}s+\cdots
\end{aligned}$$

There is no term with $1/s$ in the expansion about $s=0$, so we conclude that

$$\operatorname*{Res}_{s=0} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} = 0.$$

As a result,

$$\begin{aligned}
&\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds \\
&= 2\pi i\left\{-\frac{i}{2}e^{i\omega t}\frac{\sin\omega x}{\omega^2\cos\omega}+\frac{i}{2}e^{-i\omega t}\frac{\sin\omega x}{\omega^2\cos\omega}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{i\omega_n(\omega^2-\omega_n^2)}e^{i\omega_n t}\sin\omega_n x+\sum_{n=1}^{\infty}\frac{-(-1)^{n+1}}{i\omega_n(\omega^2-\omega_n^2)}e^{-i\omega_n t}\sin\omega_n x\right\}.
\end{aligned}$$

Divide both sides by $2\pi i$ and combine the series.

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds &= \frac{i}{2}\frac{\sin\omega x}{\omega^2\cos\omega}(-e^{i\omega t}+e^{-i\omega t}) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{i\omega_n(\omega^2-\omega_n^2)}(e^{i\omega_n t}-e^{-i\omega_n t})\sin\omega_n x \\
&= \frac{i}{2}\frac{\sin\omega x}{\omega^2\cos\omega}(-2i\sin\omega t) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{i\omega_n(\omega^2-\omega_n^2)}(2i\sin\omega_n t)\sin\omega_n x \\
&= \frac{\sin\omega x}{\omega^2\cos\omega}\sin\omega t + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\omega_n(\omega^2-\omega_n^2)}2\sin\omega_n x\sin\omega_n t
\end{aligned}$$

Therefore, the inverse Laplace transform of $F(s)$ is

$$\boxed{f(t) = \frac{\sin\omega x\sin\omega t}{\omega^2\cos\omega} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\omega_n}\cdot\frac{\sin\omega_n x\sin\omega_n t}{\omega^2-\omega_n^2},}$$

assuming that $\omega \neq \omega_n$.

**The Integral Over $C_R$**

The objective here is to show that

$$\lim_{R\to\infty}\int_{C_R} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds.$$

Start by writing hyperbolic sine and hyperbolic cosine in terms of exponential functions.

$$\begin{aligned}\int_{C_R} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds &= \int_{C_R} e^{st}\frac{\frac{e^{xs}-e^{-xs}}{2}}{s(s^2+\omega^2)\frac{e^s+e^{-s}}{2}}\,ds\\ &= \int_{C_R} e^{st}\frac{e^{(1+x)s}-e^{(1-x)s}}{s(s^2+\omega^2)(e^{2s}+1)}\,ds\end{aligned}$$

The parameterization of the semicircular arc with radius $R$ in Figure 2 is $s = Re^{i\theta}$, where $\theta$ goes from $\pi/2$ to $3\pi/2$.

$$\begin{aligned}&= \int_{\pi/2}^{3\pi/2} e^{Re^{i\theta}t}\frac{e^{(1+x)Re^{i\theta}}-e^{(1-x)Re^{i\theta}}}{Re^{i\theta}[(Re^{i\theta})^2+\omega^2](e^{2Re^{i\theta}}+1)}(Rie^{i\theta}\,d\theta)\\ &= \int_{\pi/2}^{3\pi/2} e^{R(\cos\theta+i\sin\theta)t}\frac{e^{(1+x)R(\cos\theta+i\sin\theta)}-e^{(1-x)R(\cos\theta+i\sin\theta)}}{(R^2e^{2i\theta}+\omega^2)[e^{2R(\cos\theta+i\sin\theta)}+1]}(i\,d\theta)\\ &= \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{e^{(1+x)R\cos\theta}e^{i(1+x)R\sin\theta}-e^{(1-x)R\cos\theta}e^{i(1-x)R\sin\theta}}{(R^2e^{2i\theta}+\omega^2)(e^{2R\cos\theta}e^{2iR\sin\theta}+1)}(i\,d\theta)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds\right| &= \left|\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{e^{(1+x)R\cos\theta}e^{i(1+x)R\sin\theta}-e^{(1-x)R\cos\theta}e^{i(1-x)R\sin\theta}}{(R^2e^{2i\theta}+\omega^2)(e^{2R\cos\theta}e^{2iR\sin\theta}+1)}(i\,d\theta)\right|\\ &\leq \int_{\pi/2}^{3\pi/2}\left|e^{Rt\cos\theta}e^{iRt\sin\theta}\frac{e^{(1+x)R\cos\theta}e^{i(1+x)R\sin\theta}-e^{(1-x)R\cos\theta}e^{i(1-x)R\sin\theta}}{(R^2e^{2i\theta}+\omega^2)(e^{2R\cos\theta}e^{2iR\sin\theta}+1)}(i)\right|d\theta\\ &= \int_{\pi/2}^{3\pi/2}\left|e^{Rt\cos\theta}\right|\left|e^{iRt\sin\theta}\right|\frac{\left|e^{(1+x)R\cos\theta}e^{i(1+x)R\sin\theta}-e^{(1-x)R\cos\theta}e^{i(1-x)R\sin\theta}\right|}{\left|(R^2e^{2i\theta}+\omega^2)(e^{2R\cos\theta}e^{2iR\sin\theta}+1)\right|}|i|\,d\theta\\ &= \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\frac{\left|e^{(1+x)R\cos\theta}e^{i(1+x)R\sin\theta}-e^{(1-x)R\cos\theta}e^{i(1-x)R\sin\theta}\right|}{|R^2e^{2i\theta}+\omega^2|\,|e^{2R\cos\theta}e^{2iR\sin\theta}+1|}\,d\theta\\ &\leq \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\frac{\left|e^{(1+x)R\cos\theta}e^{i(1+x)R\sin\theta}\right|+\left|e^{(1-x)R\cos\theta}e^{i(1-x)R\sin\theta}\right|}{(|R^2e^{2i\theta}|-|\omega^2|)(|e^{2R\cos\theta}e^{2iR\sin\theta}|-|1|)}\,d\theta\\ &= \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\frac{e^{(1+x)R\cos\theta}+e^{(1-x)R\cos\theta}}{(R^2-\omega^2)(e^{2R\cos\theta}-1)}\,d\theta\\ &= \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\frac{e^{(1+x)R\cos\theta}+e^{(1-x)R\cos\theta}}{(1-\frac{\omega^2}{R^2})(e^{2R\cos\theta}-1)}\,\frac{d\theta}{R^2}\end{aligned}$$

So we have

$$\left|\int_{C_R} e^{st}\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\,ds\right| \leq \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\frac{e^{(1+x)R\cos\theta}+e^{(1-x)R\cos\theta}}{(1-\frac{\omega^2}{R^2})(e^{2R\cos\theta}-1)}\,\frac{d\theta}{R^2}.$$

Take the limit of both sides now as $R \to \infty$.

$$\lim_{R\to\infty} \left| \int_{C_R} e^{st} \frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\, ds \right| \leq \lim_{R\to\infty} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta} \frac{e^{(1+x)R\cos\theta} + e^{(1-x)R\cos\theta}}{(1-\frac{\omega^2}{R^2})(e^{2R\cos\theta}-1)} \frac{d\theta}{R^2}$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R\to\infty} \left| \int_{C_R} e^{st} \frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\, ds \right| \leq \int_{\pi/2}^{3\pi/2} \lim_{R\to\infty} e^{Rt\cos\theta} \frac{e^{(1+x)R\cos\theta} + e^{(1-x)R\cos\theta}}{(1-\frac{\omega^2}{R^2})(e^{2R\cos\theta}-1)} \frac{d\theta}{R^2}$$

Since $\theta$ is between $\pi/2$ and $3\pi/2$, the cosine of $\theta$ is negative. $0 < x < 1$, $R > 0$, and $t > 0$, so all the exponential functions tend to zero, which means the integral tends to zero.

$$\lim_{R\to\infty} \left| \int_{C_R} e^{st} \frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\, ds \right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty} \left| \int_{C_R} e^{st} \frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\, ds \right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty} \int_{C_R} e^{st} \frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s}\, ds = 0.$$